Asymmetric one-dimensional exclusion processes: A two-parameter exactly solvable example

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(Received 16 February 1999; revised manuscript received 26 April 1999)

We consider a two-parameter family of asymmetric exclusion processes for particles living on a continuous one-dimensional space. Using the Bethe ansatz, the exact solution to the master equation, and from that the drift and the diffusion rate in the two particle sector, are obtained. [S1063-651X(99)09608-7]

PACS number(s): 82.20.Mj, 02.50.Ga, 05.40.−a

I. INTRODUCTION

In recent years, the asymmetric exclusion process and the problems related to it, including for example the kinetics of bipolymerization [1], dynamical models of interface growth [2], traffic models [3], the noisy Burgers equation [4], and the study of shocks [5,6], have been extensively studied. The dynamical properties of this model have been studied in [6–8]. As the results obtained by approaches like mean field are not reliable in one dimension, it is very useful to introduce solvable models and analytic methods to extract exact physical results. Among these methods is the coordinate Bethe ansatz, which was used in [9] to derive conditional probabilities for the asymmetric simple exclusion process (ASEP) on a one-dimensional lattice.

In [10], a similar technique was used to solve the drop-push model [11], and a generalized one-parameter model interpolating between the totally ASEP and the drop-push model. In this generalized process, the influence of exclusion is controlled by a parameter $\lambda\in[0,1]$, so that each particle does not necessarily stop if its right neighboring sites are occupied, but pushes these particles to the right with rates depending on the number of these particles. That is, the following process:

$$\underbrace{AA\cdots A}_{n}\emptyset\to\emptyset\underbrace{AA\cdots A}_{n}, \tag{1}$$

occur with the rate

$$r_n=\frac{1}{1+(\lambda/\mu)+\cdots+(\lambda/\mu)^n}, \tag{2}$$

where $\mu=1-\lambda$. The main idea of treating these problems is to substitute the way in which particles push each other by a suitable boundary condition, so that the master equation is the same whether there are adjacent particles or not. In [12], the same technique was applied to a two-parameter family process, which is a generalization of the process introduced in [10]. In this process, beside the parameter λ, which controls the way particles push each other, another parameter is introduced, which controls the difference of the single-particle drift rates to the right or left.

Now, as at large times the probability distribution of finding particles becomes smooth, one can substitute the master equation for the probability by an equation on a continuous space. In this way, one obtains a diffusion equation from which more compact results can be derived. So this process, which is the continuous version of asymmetric exclusion process, not only is important by itself, but also gives us correct large-time behaviors of discrete processes by simple means.

The scheme of the present paper is the following. In Sec. II, the continuous model is introduced, and compared to its discrete version, and then the evolution matrix (or the conditional probability) of the model is obtained using the Bethe ansatz.

In Sec. III the two-particle sector of the model is extensively studied. It is shown that the probability distribution of the distance between particles is independent of the two parameters introduced. (This is true only for the two-particle sector.) The diffusion-rate for the two-particle sector is then calculated, and is shown to be in agreement with the large-time result of the discrete process [12].

II. ONE-DIMENSIONAL ASYMMETRIC EXCLUSION PROCESS ON CONTINUUM

Consider the following master equation and boundary condition describing an asymmetric exclusion process:

$$\begin{aligned}\frac{\partial}{\partial t}P(x_1,x_2,\ldots,x_N;t)&=R[P(x_1-1,x_2,\ldots,x_N;t)+P(x_1,x_2-1,\ldots,x_N;t)\\&\quad+\cdots-NP(x_1,x_2,\ldots,x_N;t)]\\&\quad+L[P(x_1+1,x_2,\ldots,x_N;t)\\&\quad+P(x_1,x_2+1,\ldots,x_N;t)\\&\quad+\cdots-NP(x_1,x_2,\ldots,x_N;t)],\end{aligned} \tag{3}$$

and

$$P(\ldots,x,x,\ldots;t)=\lambda P(\ldots,x,x+1,\ldots;t)+\mu P(\ldots,x-1,x,\ldots;t), \tag{4}$$

where λ and μ must satisfy $\lambda+\mu=1$, to ensure the conservation of probability, and we have normalized the parameters R and L so that $R+L=1$. The above master equation is written for $x_i<x_{i+1}$. The above master equation and boundary condition describe a system where particles can push a

collection of n adjacent particles with rate Rr_n to the right, or with rate Ll_n to the left, where l_n is the same as r_n introduced in Eq. (2), with the roles of λ and μ interchanged [12].

Now consider a probability function varying slowly with x. This case, specially happens in large times. In this limit, one can consider the variables x_i as continuous variables and change the master equation and the boundary condition to differential equations. One obtains

$$\frac{\partial}{\partial t}P(\mathbf{x},t)=(L-R)\left(\sum_i \partial_i\right)P(\mathbf{x},t)+\frac{1}{2}\left(\sum_i \partial_i^2\right)P(\mathbf{x},t), \tag{5}$$

and

$$(\lambda\partial_{i+1}-\mu\partial_i)P_{|x_{i+1}=x_i}=0. \tag{6}$$

The master equation can still be simplified. Using a Galilean transformation: $x_i\rightarrow x_i+vt,\quad t\rightarrow t$, with $v=R-L$, one obtains

$$\frac{\partial}{\partial t}P(\mathbf{x},t)=\frac{1}{2}\nabla^2 P(\mathbf{x},t), \tag{7}$$

and the boundary condition does not change. What we have obtained is that the drifting-pushing process introduced in [12], for smooth enough probabilities is equivalent to a continuous diffusion process with a specific boundary condition. Note that Eq. (7) holds only in the physical region $x_i<x_{i+1}$.

The Bethe-ansatz solution to the master equation (7) [with the boundary condition (6)] is

$$P(\mathbf{x};t)=e^{Et}\Psi(\mathbf{x}), \tag{8}$$

where

$$\Psi(\mathbf{x})=\sum_\sigma A_\sigma e^{i\sigma(\mathbf{p})\cdot\mathbf{x}}. \tag{9}$$

The summation runs over the elements of the permutation group and A_σ's are to be determined through the boundary condition (6). Just as it was done in [12], it is seen that

$$E=-\left(\sum_j p_j^2\right)\Big/2,$$
$$A_{\sigma\sigma_i}=S(\sigma(p_i),\sigma(p_{i+1}))A_\sigma, \tag{10}$$

where σ is an arbitrary element of the permutation group and σ_i is that element which only interchanges p_i and p_{i+1}. S is an element of the scattering matrix in the two-particle sector:

$$S_{jk}:=S(p_j,p_k)=-\frac{\lambda p_k-\mu p_j}{\lambda p_j-\mu p_k}. \tag{11}$$

From these, one can obtain the conditional probability as

$$P(\mathbf{x};t|\mathbf{y};0)=\int\frac{d^Np}{(2\pi)^N}\Psi_{\mathbf{p}}(\mathbf{x})e^{E(\mathbf{p})t-i\mathbf{p}\cdot\mathbf{y}}, \tag{12}$$

where Ψ is defined in Eq. (9), and we have set $A_{\text{identity}}=1$. Also, the singularities are removed through the prescription

$$S(p_k,p_m)\rightarrow S(p_k+i\epsilon,p_m),\quad k<m, \tag{13}$$

to ensure that the conditional probability, for $\mathbf{x}$ and $\mathbf{y}$ in the physical region, enjoys the property $P(\mathbf{x};0|\mathbf{y};0)=\delta(\mathbf{x}-\mathbf{y})$.

An interesting case is when $\lambda=\mu=\frac{1}{2}$. In this case, the elements of the scattering matrix are equal to one, and we have

$$P_{\lambda=\mu}(\mathbf{x};t|\mathbf{y};0)=\sum_\sigma\frac{1}{(2\pi t)^{N/2}}e^{-(\mathbf{x}-\sigma(\mathbf{y}))^2/(2t)}. \tag{14}$$

This is just the solution to the diffusion equation with the boundary condition that the normal derivative of the function on the boundary is equal to zero, the boundary condition arrived at for $\lambda=\mu$.

III. TWO-PARTICLE SECTOR CONDITIONAL PROBABILITY, AND THE DRIFT AND DIFFUSION RATES

In the two-particle sector, one can perform the integration in Eq. (12) to obtain

$$P(\mathbf{x};t|\mathbf{y};t)=\frac{1}{2\pi t}\Bigg\{e^{-[(x_1-y_1)^2+(x_2-y_2)^2]/(2t)}+\sin 2\theta e^{-(z_1^2+z_2^2)/(2t)}+\cos 2\theta\sqrt{\frac{\pi}{2t}}z_1' e^{-z_1'^2/(2t)}\times\left[-1+\text{erf}\left(\frac{z_2'}{\sqrt{2t}}\right)\right]\Bigg\}, \tag{15}$$

where we have defined

$$z_1:=x_1-y_2,$$
$$z_2:=x_2-y_1,$$
$$\tan\theta:=\lambda/\mu, \tag{16}$$
$$z_1':=z_1\cos\theta+z_2\sin\theta,$$
$$z_2':=-z_1\sin\theta+z_2\cos\theta.$$

To obtain the drift and diffusion rates, let us find the probability density of finding two particles at a distance x. We have

$$P_r(x;t|y_1,y_2;0):=\int_{-\infty}^{+\infty}dx_2\,P(x_2-x,x_2;t|y_1,y_2,0) \tag{17}$$

and, using Eq. (12), we arrive at

$$P_r(x;t|y,0)=\frac{1}{\sqrt{4\pi t}}\{e^{-(x-y)^2/4t}+e^{-(x+y)^2/4t}\}, \tag{18}$$

where we have defined $y:=y_2-y_1$. To proceed, we must calculate the probability densities of finding the first and second particles at x:

$$P_1(x)=\int_x^{+\infty} dx_2\, P(x,x_2) \tag{19}$$

and

$$P_2(x)=\int_{-\infty}^{x} dx_1\, P(x_1,x). \tag{20}$$

Starting from the master equation (7), and the boundary condition (6), the time evolutions for P_1 and P_2 can be written as

$$\dot{P}_1(x)=\frac{1}{2}\partial^2 P_1(x)+\lambda\partial P(x,x) \tag{21}$$

and

$$\dot{P}_2(x)=\frac{1}{2}\partial^2 P_2(x)-\mu\partial P(x,x). \tag{22}$$

From these, we arrive at

$$\frac{d\langle r\rangle}{dt}:=\frac{d}{dt}(\langle x_2\rangle-\langle x_1\rangle)=P_r(0) \tag{23}$$

and

$$\frac{d\langle X\rangle}{dt}:=\frac{1}{2}\frac{d}{dt}(\langle x_2\rangle+\langle x_1\rangle)=\frac{\mu-\lambda}{2}P_r(0)=\frac{\mu-\lambda}{2}\frac{d\langle r\rangle}{dt}. \tag{24}$$

So,

$$\langle X\rangle=\langle X\rangle_0+\frac{\mu-\lambda}{2}(\langle r\rangle-\langle r\rangle_0). \tag{25}$$

The asymptotic behavior P_r is, from Eq. (18),

$$P_r(x)=\frac{1}{\sqrt{\pi t}}+O(t^{-3/2}), \tag{26}$$

from which we obtain

$$\begin{aligned}\langle r\rangle&=C+2\sqrt{t/\pi}+O(t^{-1/2}),\\ \langle X\rangle&=\langle X\rangle_0+(\mu-\lambda)(C+2\sqrt{t/\pi}-\langle r\rangle_0)/2+O(t^{-1/2}).\end{aligned} \tag{27}$$

Here C is a constant depending on the initial conditions. The drift rate for large times is now easily obtained:

$$V:=\lim_{t\to\infty}\frac{d\langle X\rangle}{dt}=0. \tag{28}$$

The diffusion rate is

$$\frac{d(\langle X^2\rangle-\langle X\rangle^2)}{dt}=\frac{1}{2}\frac{d}{dt}(\langle x_1^2\rangle+\langle x_2^2\rangle)-\frac{d}{dt}(\langle X\rangle^2), \tag{29}$$

where

$$\langle X^2\rangle:=\frac{1}{2}\langle x_1^2+x_2^2\rangle. \tag{30}$$

Performing the calculations, one obtains

$$\frac{d(\langle X^2\rangle-\langle X\rangle^2)}{dt}=1+\frac{(\mu-\lambda)^2}{2}\left[1-\mathrm{erf}\left(\frac{y}{\sqrt{4t}}\right)-(\langle r\rangle-y)\frac{e^{-y^2/(4t)}}{\sqrt{\pi t}}\right]. \tag{31}$$

Then, using the asymptotic behavior of the error function, it is seen that

$$\Delta:=\lim_{t\to\infty}\frac{d(\langle X^2\rangle-\langle X\rangle^2)}{dt}=1+(\mu-\lambda)^2\left(\frac{1}{2}-\frac{1}{\pi}\right). \tag{32}$$

It is seen that the results (28) and (32) coincide with the corresponding results of [12]. These results also conicide with those obtained in [9]. To see this correspondence, however, one should note that in [9], there is just one parameter, namely D_R (which corresponds to R and μ both), and that we have used a Galilean transformation to make $V=0$.

ACKNOWLEDGMENTS

M.K. would like to thank M. Alimohammadi and V. Karimipour for stimulating discussions.

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